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ON THE MATHEMATICAL STRUCTURE OF T.D. LEE'S MODEL OF A RENORMALIZABLE FIELD THEORY

BY

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It is shown that the appropriate mathematical formalism of the field theoretical model recently proposed by T. D. LEE must use an indefinite metric to describe the norm of the state vector in the Hilbert space. The appearance of the indefinite metric is intimately connected with a new state of the V-particle having an energy that is below the mass of the "normal" V-particle. It is further shown that the S-matrix for this model is not unitary and that the probability for an incoming V-particle in the normal state and a boson, to make a transition to an outgoing V-particle in the new state and another boson, must be negative if the sum of all transition probabilities for the incoming state mentioned shall add up to one.

Introduction.

In a recent paper[1)], T. D. LEE has suggested a very interesting model of a renormalizable field theory. This model is simple enough to allow a more or less explicit solution, but complicated enough to contain many features characteristic of more realistic theories. It uses not only a renormalization of the mass of one kind of particles involved, but also a renormalization of the coupling constant g describing the interaction between the particles. In the explicit solution found by LEE, the ratio between the square of the renormalized coupling constant g and the square of the unrenormalized coupling constant g_0 is given by an expression of the form

$$\frac{g^2}{g_0^2} = 1 - A \cdot g^2, \tag{1}$$

where A is a divergent integral. The ratio (1) is thus equal to $-\infty$. This is a very remarkable result, as according to very general principles[2)], this ratio should lie between one and zero. It is the aim of the present note to investigate the mathematical origin of the result (1) and to show that the violation of general principles implied by (1) also has observable consequences insofar as the S-matrix of the theory turns out not to be unitary.

To avoid the manipulation of divergent integrals we introduce a cut-off factor in the interaction. It will then appear that abnormal values of the ratio (1) are also obtained for a finite value of the cut-off and are not immediately connected with the infinities in the original formulation. To make our discussion reasonably self-contained we start with a survey of the foundations of the Lee model and with an outline of the way in which the renormalizations have to be performed in this case.

I. Renormalization of the Lee Model.

Let us consider a system with three different kinds of particles which, following LEE, we call V-particles, N-particles, and θ-particles. To each kind of particles corresponds a field that will be denoted by ψ_V, ψ_N, and a, respectively. The system is governed by the following *unrenormalized* Hamiltonian:

$$H = H_0 + H_{\mathrm{int}}, \tag{2}$$

$$\left.\begin{aligned} H_0 = & \sum_{\bar{p}} E_V(\bar{p})\,\psi_V^*(\bar{p})\,\psi_V(\bar{p}) + \sum_{\bar{p}} E_N(\bar{p})\,\psi_N^*(\bar{p})\,\psi_N(\bar{p}) \\ & + \sum_{\bar{k}} \omega(\bar{k})\,a^*(\bar{k})\,a(\bar{k}), \end{aligned}\right\} \tag{3}$$

$$H_{\mathrm{int}} = -\frac{g_0}{\sqrt{V}} \sum_{\bar{p}=\bar{p}'+\bar{k}} \frac{f(\omega)}{\sqrt{2\,\omega}} \left(\psi_V^*(\bar{p})\,\psi_N(\bar{p}')\,a(\bar{k}) + a^*(\bar{k})\,\psi_N^*(\bar{p}')\,\psi_V(\bar{p})\right). \tag{4}$$

The operators in (3) and (4) can be thought of as being written in p-space and in a Schrödinger representation. The model does not have invariance with respect to the Lorentz group and it will not be necessary to use the more sophisticated representations of relativistic field theories. The energies $E_V(\bar{p})$, $E_N(\bar{p})$, and $\omega(\bar{k})$ are, in principle, arbitrary functions of the momenta involved and the theory can be treated for any form of these functions. However, for our purpose, it will be sufficient to consider the following special case,

$$E_V(\bar{p}) = E_N(\bar{p}) = m \quad (\text{independent of } \bar{p}), \tag{5}$$

$$\omega(\bar{k}) = \sqrt{\bar{k}^2 + \mu^2}. \tag{6}$$

In particular, *Eq.* (5) will simplify the formal expressions to some extent without interfering with the interesting features of the result. If one wishes, this choice of the energies as functions of the momenta can be thought of as giving a model for the interaction of very heavy V- and N-particles (with equal masses) with light, relativistic θ-particles. The function $f(\omega)$ in (4) is the cut-off function mentioned earlier and is introduced to make the sums, appearing later, convergent. The quantity V is the volume of periodicity.

The field operators obey the following commutation and anti-commutation relations:

$$\{\psi_V^*(\bar{p}),\, \psi_V(\bar{p}')\} = \{\psi_N^*(\bar{p}),\, \psi_N(\bar{p}')\} = \delta_{\bar{p},\,\bar{p}'}, \tag{7}$$

$$\{\psi_V(\bar{p}),\, \psi_V(\bar{p}')\} = \{\psi_V(\bar{p}),\, \psi_N(\bar{p}')\} = \cdots\cdots = 0, \tag{8}$$

$$[a(\bar{k}),\, a^*(\bar{k}')] = \delta_{\bar{k},\,\bar{k}'}, \tag{9}$$

$$[a(\bar{k}),\, \psi_V(\bar{p})] = [a(\bar{k}),\, \psi_N(\bar{p}')] = \cdots\cdots = 0. \tag{10}$$

With the aid of these commutators we can set up a representation in the Hilbert space, where each state is characterized by the number of particles present. Further, each state in this representation is an eigenstate of the free-particle Hamiltonian H_0 in (3), but not of the total Hamiltonian (2). Let us denote these states by

$$\left| n_V,\, n_N,\, n_k \right\rangle, \tag{11}$$

where n_V, n_N, and n_k are the numbers of "free" V-particles, N-particles, and θ-particles present[3)].

With the aid of (7)—(10) it can easily be verified that the following two operators commute with the total Hamiltonian.

$$Q_1 = \sum_{\bar{p}} \psi_V^*(\bar{p})\,\psi_V(\bar{p}) + \sum_{\bar{p}} \psi_N^*(\bar{p})\,\psi_N(\bar{p}), \tag{12}$$

$$Q_2 = \sum_{\bar{p}} \psi_N^*(\bar{p})\,\psi_N(\bar{p}) - \sum_{\bar{k}} a^*(\bar{k})\,a(\bar{k}), \tag{13}$$

$$[H,\, Q_i] = 0, \quad i = 1,2. \tag{14}$$

As each state (11) is also an eigenstate of the operators Q_i, it follows that the eigenstates of the total Hamiltonian H can be built up as linear combinations of states (11) belonging to the same eigenvalue q_i. This will considerably simplify the problem of diagonalizing the total Hamiltonian and, in some cases, even give an explicit solution. As an example, we may mention that there is only one of the states (11) which has $q_1 = q_2 = 0$, *viz.* the state $|0, 0, 0\rangle$ or the "free-particle vacuum". Hence, this state is also an eigenstate of the total Hamiltonian, and a simple calculation gives the eigenvalue zero for this operator. The "physical vacuum" is thus the same as the free-particle vacuum for this model. In the same way, we can show that the physical N-particle states and the physical θ-particle states are identical with the corresponding free-particle states, but that the free V-particle states are *not* eigenstates of the total Hamiltonian. It will be necessary to consider a linear combination of the states $|1_V, 0, 0\rangle$ and $|0, 1_N, 1_k\rangle$ to construct an eigenstate of the total Hamiltonian for this case. We shall later return to this point. For the moment we only remark that, under these circumstances, it will not be necessary to introduce renormalizations of the masses of the N-particles or the θ-particles. The mass renormalization in the model is now performed by adding the following term to the Hamiltonian (this term will *not* change the conservation equations (14)):

$$\delta H = -\delta m \sum_{\bar{p}} \psi_V^*(\bar{p})\, \psi_V(\bar{p}). \tag{15}$$

The constant δm in (15) should, if possible, be determined in such a way that the state corresponding to the physical V-particle has the mass m appearing in H_0. Following the custom in quantum electrodynamics, we also introduce a renormalization of the coupling constant g_0 and of the field operator ψ_V by a factor N in the following way:

$$g = g_0 \cdot N, \tag{16}$$

$$\psi'_V(\bar{p}) = \psi_V(\bar{p}) \frac{1}{N}. \tag{17}$$

It is important to realize that the constant N in (16) and (17) can by definition be chosen to be real, as there is always an arbitrary phase factor in the field operators. The choice of a real N only fixes the phase connection between ψ_V and ψ'_V and can have no physical consequences. The value of N is determined by the condition[4)]

$$\langle 0 \,|\, \psi'_V(\bar{p}) \,|\, V \rangle = 1. \tag{18}$$

The state $|\,V\rangle$ in (18) is the physical V-particle state and the state $|\,0\rangle$ the physical vacuum. In what follows, we drop the dash on the renormalized ψ_V-operator as the corresponding unrenormalized operator will not be used again. In terms of our renormalized quantities the Hamiltonian and the canonical commutators will now read

$$H = H_0 + H_{\text{int}} + \delta H, \tag{19}$$

$$H_0 = mN^2 \sum_{\bar{p}} \psi_V^*(\bar{p})\,\psi_V(\bar{p}) + m \sum_{\bar{p}} \psi_N^*(\bar{p})\,\psi_N(\bar{p}) + \sum_{\bar{k}} \omega(\bar{k})\,a^*(\bar{k})\,a(\bar{k}), \tag{20}$$

$$H_{\text{int}} = -\frac{g}{\sqrt{V}} \sum_{\bar{p}=\bar{p}'+\bar{k}} \frac{f(\omega)}{\sqrt{2\,\omega}} \left(\psi_V^*(\bar{p})\,\psi_N(\bar{p}')\,a(\bar{k}) + a^*(\bar{k})\,\psi_N^*(\bar{p}')\,\psi_V(\bar{p})\right), \tag{21}$$

$$\delta H = -\,\delta m\, N^2 \sum_{\bar{p}} \psi_V^*(\bar{p})\,\psi_V(\bar{p}), \tag{22}$$

$$\{\psi_V^*(\bar{p}),\ \psi_V(\bar{p}')\} = \frac{1}{N^2}\,\delta_{\bar{p},\bar{p}'} \quad \text{(other commutators unchanged).} \tag{23}$$

Eqs. (19)—(23) will be the foundation for the following discussion.

II. The Physical *V*-Particle States and the States Describing the Scattering of one *N*-Particle and one θ-Particle.

We now try to find an eigenstate of the total Hamiltonian of the form

$$|\,z\rangle = |\,1_V, 0, 0\rangle + \sum_{\bar{k}} \Phi(\bar{k})\,|\,0, 1_N, 1_k\rangle. \tag{24}$$

In this expression all terms have the same total momentum. In the following formulae, a factor expressing conservation of three-dimensional momentum is very often left out. Calling the eigenvalue of the state (24) $m + \omega_0$, and using (19)—(23), we obtain after some straightforward calculations

$$\omega_0 + \delta m = -\frac{g}{N\cdot\sqrt{V}}\sum_{\bar{k}}\frac{\Phi(\bar{k})\,f(\omega)}{\sqrt{2\,\omega}}, \tag{25}$$

$$(\omega - \omega_0)\,\Phi(\bar{k}) = \frac{g}{N\sqrt{V}}\frac{f(\omega)}{\sqrt{2\,\omega}}. \tag{26}$$

Eliminating $\Phi(\bar{k})$ from (25) and (26) we get the following equation for the determination of the eigenvalue ω_0:

$$\omega_0 + \delta m + \frac{g^2}{2\,N^2\,V}\sum_{\bar{k}}\frac{f^2(\omega)}{\omega}\frac{1}{\omega - \omega_0} = 0\,. \tag{27}$$

The constant δm is now determined from the condition that $\omega_0 = 0$ should be one solution of (27). The corresponding eigenstate (24) is, when properly normalized, the physical V-particle state. This gives us

$$\delta m = \frac{-g^2}{2\,V}\frac{1}{N^2}\sum_{\bar{k}}\frac{f^2(\omega)}{\omega^2}, \tag{28}$$

$$|\,V\rangle = C\left[\,|\,1_V,\,0,\,0\,\rangle + \frac{g}{N\sqrt{2\,V}}\sum_{\bar{k}}\frac{f(\omega)}{\omega^{3/2}}\,|\,0,\,1_N,\,1_k\,\rangle\right], \tag{29}$$

$$C^{-2} = 1 + \frac{g^2}{2\,VN^2}\sum_{\bar{k}}\frac{f^2(\omega)}{\omega^3}. \tag{30}$$

Furthermore, using *Eq.* (18), we get

$$C = N \tag{31}$$

or

$$|\,V\rangle = N\,|\,1_V,\,0,\,0\,\rangle + \frac{g}{\sqrt{2\,V}}\sum_{\bar{k}}\frac{f(\omega)}{\omega^{3/2}}\,|\,0,\,1_N,\,1_k\,\rangle, \tag{32}$$

$$N^2 = 1 - \frac{g^2}{2\,V}\sum_{\bar{k}}\frac{f^2(\omega)}{\omega^3}. \tag{33}$$

The results obtained so far in this paragraph correspond exactly to those obtained by LEE. In particular, *Eqs.* (33) and (16) together give LEE's result (1) if the form factor is put equal to unity for all values of ω. However, if we have a finite cut-off, *Eq.* (33) can be written

$$N^2 = 1 - \frac{g^2}{g_{\text{crit}}^2}, \tag{34}$$

$$g_{\text{crit}}^{-2} = \frac{1}{2V} \sum_{\bar{k}}' \frac{f^2(\omega)}{\omega^3}. \tag{34 a}$$

The value (34) of N^2 lies between zero and one, as was to be expected, only if the renormalized coupling constant g is less than a critical value g_{crit} depending on the cut-off function and defined by (34 a). If there is no cut-off, the critical value of the coupling is zero. Further, if the renormalization of the coupling constant is not performed explicitly, but if all quantities are expressed in terms of the original constant g_0, we have to substitute the expression

$$g^2 = \frac{g_0^2 \cdot g_{\text{crit}}^2}{g_0^2 + g_{\text{crit}}^2} \tag{35}$$

for g^2 everywhere in our formulae above. *Eq.* (35) contains the definite prediction that the renormalized coupling is always less than the critical coupling if the Hamiltonian is hermitian, *i.e.* if g_0 is real. As stressed by LEE, it is of some interest to investigate also the case of the renormalized coupling being larger than the critical value and the Hamiltonian being non-hermitian. The crucial question to be answered is whether this violation of the ordinary methods of quantum mechanics will have any observable consequences or if we are able in this way to get an at least partially satisfactory theory.

We now turn to the investigation of the other solutions to the eigenvalue problem (27). Making use of (28) and (33) we can rewrite *Eq.* (27) in the following way:

$$h(\omega_0) \equiv \omega_0 \left[1 + \frac{g^2}{2V} \sum_{\bar{k}}' \frac{f^2(\omega)\,\omega_0}{\omega^3(\omega - \omega_0)} \right] = 0. \tag{36}$$

The second factor in (36) has a pole each time $\omega_0 = \omega_i$, where ω_i is an eigenvalue of the unperturbed Hamiltonian H_0. As the derivative of the last factor in (36) with respect to ω_0 is always positive, this factor must vanish once, and only once, in each interval (ω_i, ω_{i+1}). The corresponding eigenstates (24) describe the scattering of one N-particle and one θ-particle. After some formal manipulations these states can be written,

$$|N, \theta\rangle = |0, 1_N, 1_k\rangle + \sum_{\bar{k}'} \alpha(\bar{k}, \bar{k}')|0, 1_{N'}, 1_{k'}\rangle + \beta(\bar{k}) N |1_V, 0, 0\rangle, \tag{37}$$

$$\alpha(\bar{k}, \bar{k}') = \frac{g}{\sqrt{2V}} \frac{\beta(\bar{k}) f(\omega')}{\sqrt{\omega'}} \left\{ P \frac{1}{\omega' - \omega} + i\pi\delta(\omega' - \omega) \right\}, \tag{38}$$

$$\beta(\bar{k}) = -\frac{g f(\omega)}{\sqrt{2V}\,\omega^{3/2}} \left[1 + \frac{g^2 \omega}{2V} \sum_{\bar{k}'} \frac{f^2(\omega')}{\omega'^3} \left(P \frac{1}{\omega' - \omega} + i\pi\delta(\omega' - \omega) \right) \right]^{-1}. \tag{39}$$

In (38) and (39), the limit $V \to \infty$ has been anticipated and these equations contain a prescription how the denominators must be treated when the integration over $\bar{k}'$ is performed. This prescription corresponds to only outgoing waves in the second term of (37). The only incoming particles in these states have momentum $\bar{k}$. From the formulae above it is possible to compute that part of the S-matrix which corresponds to the scattering of N-particles and θ-particles by each other. The result is the unitary matrix

$$\langle N, \theta | S | N', \theta' \rangle = \delta_{\bar{k}, \bar{k}'} + \frac{i\pi g^2}{V} \frac{f^2(\omega)}{\omega} \frac{\delta(\omega' - \omega)}{h(\omega) + i \frac{g^2}{4\pi} |\bar{k}| f^2(\omega)}. \tag{40}$$

From (40) we get the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{|\bar{k}|^2} \sin^2 \delta \tag{41}$$

with

$$\operatorname{tg} \delta = \frac{g^2}{4\pi} \frac{|\bar{k}| f^2(\omega)}{h(\omega)}. \tag{42}$$

Again, this corresponds exactly to the results obtained by Lee. In the last three formulae, the limit $V \to \infty$ is performed and the integral appearing in $h(\omega)$ (*Eq.* (36)) is defined to be a principal value.

It remains to discuss the important question whether the states (32) and (37) obtained so far form a complete set or if *there are possibly other states of the form* (24) *which are also eigenstates of the total Hamiltonian.* If other states exist, they must correspond to other solutions of the eigenvalue problem (36). We therefore begin by a more detailed discussion of this equation. The argument given so far has exhausted all roots of this equation in the domain $\omega_0 > \mu$*. For $\omega_0 < \mu$, we find that the second factor of (36) still has a positive derivative and that it approaches the value $N^2 = 1 - g^2/g^2_{\text{crit}}$ for very large values of $|\omega_0|$. If the coupling constant is less than the critical value, we have no extra root of (36) and the states considered so far form a complete set. On the other hand, *if the coupling is larger than the critical coupling, there will be exactly one extra root of* (36) *for* $\omega_0 < \mu$. The corresponding eigenstate is not a scattering state, but will represent another state of the V-particle.† This state can be constructed explicitly from the formalism given here, and the result is

$$|V_{-\lambda}\rangle = \frac{1}{\sqrt{|h'(-\lambda)|}} \left[N \cdot |1_V, 0, 0\rangle + \frac{g}{\sqrt{2V}} \sum_{\overline{k}} \frac{f(\omega)}{\sqrt{\omega}} \frac{1}{\omega + \lambda} |0, 1_N, 1_k\rangle \right], \quad (43)$$

$$h(-\lambda) = 0; \quad \lambda > 0. \quad (44)$$

The normalization of the state (43) is chosen in a way that will be justified in the next paragraph.

It will be shown in Appendix I that *Eq.* (36) has no non-real roots.

* If the cut-off function vanishes exactly for ω larger than some value Ω, the domain $\omega_0 > \Omega$ needs a special discussion, as the argument after *Eq.* (36) will not be valid there. Actually, it can be shown that there is an extra root in this domain if g is less than the critical value g_{crit}. To avoid inessential complications of the argument, we therefore consider only cut-off functions that have a long tail as, *e. g.*, $f(\omega) = e^{-\omega/\Omega}$, where this question will not appear.

† In footnote 4 of Lee's paper, the possibility of another stable state of the V-particle is briefly mentioned, but no detailed investigation of its properties is given. In our discussion, this state will be of paramount importance.

III. Introduction of an Indefinite Metric in the Hilbert Space.

The negative sign for N^2 in (34), if g is larger than g_{crit}, obviously leads to difficulties with the normalization of the physical V-particle state (32). If we try to correct the normalization of this state by multiplying it with a suitable factor, we are ultimately led to a modification of our renormalization prescriptions insofar as we can no longer use the same factor in (16) and (17) to renormalize the coupling constant and the field operator ψ_V. In this case, extra factors have to be inserted in the interaction Hamiltonian (21), and it can easily be seen that it is not possible in this way to make the theory mathematically consistent. The only possibility of saving the normalization of the state (32) is then to *define* the norm of a state $\alpha \left| n_V, n_N, n_k \right\rangle$ to be $|\alpha|^2(-1)^{n_V}$. As N^2 in our case is real and negative, this indefinite metric will be the appropriate mathematical framework for the Lee model.[5)] The introduction of this device will not change many of the formal operations performed earlier, and particularly the scattering states (37) and the S-matrix (40) will be uninfluenced by it. On the other hand, the norm of the state (32) will be one as it stands in the new metric. The norm of the state (43) will be

$$\left.\begin{aligned} &\frac{1}{|h'(-\lambda)|}\left[N^2 + \frac{g^2}{2V}\sum_{\bar{k}} \frac{f^2(\omega)}{\omega(\omega+\lambda)^2}\right] \\ &= \frac{1}{|h'(-\lambda)|}\left[1 + \frac{g^2}{2V}\sum_{\bar{k}} \frac{f^2(\omega)}{\omega}\left[\frac{1}{(\omega+\lambda)^2} - \frac{1}{\omega^2}\right]\right] \\ &= \frac{1}{|h'(-\lambda)|}\frac{g^2}{2V}\sum_{\bar{k}} \frac{f^2(\omega)}{\omega}\left[\frac{1}{(\omega+\lambda)^2} - \frac{1}{\omega^2} + \frac{\lambda}{\omega^2(\omega+\lambda)}\right] = \frac{h'(-\lambda)}{|h'(-\lambda)|} = -1. \end{aligned}\right\} \tag{45}$$

The norm of the state $\left| V_{-\lambda} \right\rangle$ *is negative* and has been normalized to -1 in (43).

To make the formal discussion as simple as possible it will now be convenient to introduce a "metric operator" η[5)] which has the following matrix elements for the free-particle states (11):

$$\langle n_V, n_N, n_k | \eta | n'_V, n'_N, n'_k \rangle = \delta_{n_V n'_V} \cdot \delta_{n_N n'_N} \cdot \delta_{n_k n'_k} \cdot (-1)^{n_V}. \tag{46}$$

For the physical states considered up till now, we have

$$\langle V | \eta | V \rangle = \langle N, \theta | \eta | N, \theta \rangle = 1, \tag{47}$$

$$\langle V_{-\lambda} | \eta | V_{-\lambda} \rangle = -1. \tag{48}$$

The non-diagonal elements of η between these states are all zero. The condition for an operator F to have real expectation values is no longer that it is hermitian, but rather that it is "self-adjoint" in the following sense:

$$F = F^{+} \equiv \eta F^{*} \eta. \tag{49}$$

A detailed examination of the foregoing calculations shows that the introduction of the indefinite metric will make the mathematics formally consistent if the adjoint operators ψ_V^+, ψ_N^+, and a^+ are introduced in *Eqs.* (20)—(23) instead of the operators ψ_V^*, ψ_N^*, and a^*. This will make the Hamiltonian self-adjoint. On the other hand, the right-hand side of (23) will no longer have a definite sign, and a negative value of this *c*-number will not necessarily be inconsistent with the foundations of the theory. A special case of the expectation value of this anticommutator is examined in Appendix I.

If the transformation leading from the free particle states $| n \rangle$ to the physical states $| P \rangle$ is written as a matrix U,

$$| P \rangle = \sum_{| n \rangle} | n \rangle \langle n | U | P \rangle, \tag{50}$$

this matrix will not be unitary, but have the property

$$U^{+} U = \eta U^{*} \eta \, U = 1. \tag{51}$$

It is then important to decide whether the *S*-matrix of the theory also has the property (51) rather than being unitary. This expectation is not in contradiction with the result (40), as the operator η has only matrix elements $+1$ for the physical states involved there. *Eq.* (51) will have non-trivial consequences only if *physical* states with a non-positive norm are involved. The simplest process of this kind is the scattering of a θ-particle by a *V*-particle either in its normal state or in the state $| V_{-\lambda} \rangle$. In the former case, it is to be expected that transitions of the V-particle to its new state take place and that these transitions possibly occur with "negative probabilities". The following paragraph is devoted to a discussion of these problems.

IV. The Scattering of θ-Particles by V-Particles.

We will now study eigenvectors of the total Hamiltonian of the form

$$|z\rangle = \sum_{\bar{k}} \Phi_1(\bar{k}) \cdot N \cdot |1_V, 0, 1_k\rangle + \sum_{\bar{k}, \bar{k}'} \Phi_2(\bar{k}, \bar{k}') |0, 1_N, 1_k, 1_{k'}\rangle. \quad (52)$$

If the eigenvalue is again called $m + \omega_0$, a straightforward calculation will yield the following equations for the coefficients in (52):

$$\Phi_1(\bar{k})(\omega - \omega_0 - \delta m) = \frac{1}{N^2} g \sqrt{\frac{2}{V}} \sum_{\bar{k}'} \Phi_2(\bar{k}, \bar{k}') \frac{f(\omega')}{\sqrt{\omega'}}, \quad (53)$$

$$\Phi_2(\bar{k}, \bar{k}')(\omega + \omega' - \omega_0) = \frac{g}{\sqrt{2V}} \cdot \frac{1}{2} \left[\Phi_1(\bar{k}) \frac{f(\omega')}{\sqrt{\omega'}} + \Phi_1(\bar{k}') \frac{f(\omega)}{\sqrt{\omega}} \right]. \quad (54)$$

In this case, we are not interested in the complete set of states (52), but will only try to find those special states corresponding to the scattering of a θ-particle by a V-particle in its normal state. In other words, we look for solutions to (53) and (54) where $\Phi_1(\bar{k})$ is of the form

$$\Phi_1(\bar{k}, \bar{k}_0) = \delta_{\bar{k}, \bar{k}_0} + \psi(\bar{k}, \bar{k}_0) \quad (55)$$

with outgoing waves only in $\psi(\bar{k}, \bar{k}_0)$ and in $\Phi_2(\bar{k}, \bar{k}')$. The last condition gives us

$$\left. \begin{aligned} \Phi_2(\bar{k}, \bar{k}', \bar{k}_0) &= \frac{g}{\sqrt{2V}} \cdot \frac{1}{2} \cdot \left[\Phi_1(\bar{k}, \bar{k}_0) \frac{f(\omega')}{\sqrt{\omega'}} \right. \\ &\left. + \Phi_1(\bar{k}', \bar{k}_0) \frac{f(\omega)}{\sqrt{\omega}} \right] \left[P \frac{1}{\omega + \omega' - \omega_0} + i\pi\delta(\omega + \omega' - \omega_0) \right] \end{aligned} \right\} \quad (56)$$

or, using (28) and (33),

$$\left.\begin{aligned}&\Phi_1(\bar{k},\bar{k}_0)\,h(\omega_0-\omega)\\ =\frac{g^2}{2V}\frac{f(\omega)}{\sqrt{\omega}}&\sum_{\bar{k}'}\frac{f(\omega')\,\Phi_1(\bar{k}',\bar{k}_0)}{\sqrt{\omega'}}\left[P\frac{1}{\omega+\omega'-\omega_0}+i\pi\delta(\omega+\omega'-\omega_0)\right].\end{aligned}\right\}\quad(57)$$

Contrary to the situation in paragraph II, it will not be possible to find an explicit solution to *Eq.* (57). However, this will not be necessary for our purpose, as it is sufficient here to investigate the properties of the S-matrix. This can be done with a method very similar to Møller's proof of the unitarity of the S-matrix if the Hamiltonian is hermitian.[6)] Following Møller, we introduce the following quantity

$$\left.\begin{aligned}&U(\bar{k},\bar{k}_0)\\ =i\frac{g^2}{2V}\frac{f(\omega)}{\sqrt{\omega}}&\sum_{\bar{k}'}\frac{f(\omega')\,\Phi_1(\bar{k}',\bar{k}_0)}{\sqrt{\omega'}}\left[P\frac{1}{\omega+\omega'-\omega_0}+i\pi\delta(\omega+\omega'-\omega_0)\right].\end{aligned}\right\}\quad(58)$$

From *Eq.* (57) we then conclude

$$\left.\begin{aligned}\sum_{\bar{k}}\Phi_1^*(\bar{k},\bar{k}_0)\,U(\bar{k},\bar{k}_0')=i\frac{g^2}{2V}&\sum_{\bar{k},\bar{k}''}\frac{\Phi_1^*(\bar{k},\bar{k}_0)\,f(\omega)}{\sqrt{\omega}}\frac{f(\omega'')\,\Phi_1(\bar{k}'',\bar{k}_0')}{\sqrt{\omega''}}\\ &\times\left[P\frac{1}{\omega+\omega''-\omega_0'}+i\pi\delta(\omega+\omega''-\omega_0')\right],\end{aligned}\right\}\quad(59)$$

$$\left.\begin{aligned}\sum_{\bar{k}''}U^*(\bar{k}'',\bar{k}_0)\,\Phi_1(\bar{k}'',\bar{k}_0')=-i\frac{g^2}{2V}&\sum_{\bar{k},\bar{k}''}\frac{\Phi_1^*(\bar{k},\bar{k}_0)\,f(\omega)}{\sqrt{\omega}}\frac{f(\omega'')\,\Phi_1(\bar{k}'',\bar{k}_0')}{\sqrt{\omega''}}\\ &\times\left[P\frac{1}{\omega+\omega''-\omega_0}-i\pi\delta(\omega+\omega''-\omega_0)\right].\end{aligned}\right\}\quad(60)$$

The sum of (59) and (60) vanishes, as does the corresponding sum in Møller's paper, only if $\omega_0 < 2\mu$. In this case, $\omega+\omega''-\omega_0$ never vanishes in the physical interval (μ,∞) of the frequencies ω, ω'', and the transition $V+\theta\rightarrow N+\theta'+\theta''$ cannot occur on the energy shell. In the opposite case, $\omega_0 > 2\mu$, this transition causes a slight complication and we get

$$\left.\begin{aligned}&\delta(\omega_0-\omega_0')\Big[\sum_{\bar k}\Phi_1^*(\bar k,\bar k_0)\,U(\bar k,\bar k_0')+\sum_{\bar k}U^*(\bar k,\bar k_0)\,\Phi_1(\bar k,\bar k_0')\Big]\\&=-\frac{\pi g^2}{V}\delta(\omega_0-\omega_0')\sum_{\bar k,\bar k''}\Phi_1^*(\bar k,\bar k_0)\frac{f(\omega)f(\omega'')}{\sqrt{\omega\omega''}}\Phi_1(\bar k'',\bar k_0')\,\delta(\omega+\omega''-\omega_0).\end{aligned}\right\}\quad(61)$$

With the aid of (55), (57), (58), and the vanishing of $h(0)$, we have

$$\psi(\bar k,\bar k_0)\,h(\omega_0-\omega)=i\,U(\bar k,\bar k_0).\qquad(62)$$

We write the solution of (62) symbolically as

$$\psi(\bar k,\bar k_0)=i\,\frac{U(\bar k,\bar k_0)}{h(\omega_0-\omega)_+},\qquad(63)$$

where the plus sign indicates that outgoing waves are to be chosen at the zeros of $h(\omega_0-\omega)$. Using this result, we can write (61) as

$$\left.\begin{aligned}&\delta(\omega_0-\omega_0')\,[U(\bar k_0,\bar k_0')+U^*(\bar k_0',\bar k_0)]\\&+i\delta(\omega_0-\omega_0')\sum_{\bar k}U^*(\bar k,\bar k_0)\,U(\bar k,\bar k_0')\left[\frac{1}{h(\omega_0-\omega)_+}-\frac{1}{h(\omega_0-\omega)_-}\right]\\&+\frac{\pi g^2}{V}\delta(\omega_0-\omega_0')\sum_{\bar k,\bar k''}\Phi_1^*(\bar k,\bar k_0)\frac{f(\omega)f(\omega'')}{\sqrt{\omega\omega''}}\Phi_1(\bar k'',\bar k_0')\,\delta(\omega+\omega''-\omega_0)=0.\end{aligned}\right\}\quad(64)$$

The second bracket of (64) can be rewritten in the following way:

$$\frac{1}{h(\omega_0-\omega)_+}-\frac{1}{h(\omega_0-\omega)_-}=-2\pi i\sum_{\varrho_i}\frac{1}{h'(\varrho_i)}\delta(\omega_0-\omega-\varrho_i),\qquad(65)$$

where the summation is over all the roots of the equation $h(x)=0$. To simplify the notations further, we introduce the matrices

$$\langle V,\theta\,|\,R^{(1)}\,|\,V',\theta'\rangle=2\pi\delta(\omega-\omega')\,U(\bar k,\bar k'),\qquad(66)$$

$$\langle V_{-\lambda},\theta\,|\,R^{(2)}\,|\,V,\theta'\rangle=2\pi\delta(\omega+\lambda-\omega')\frac{U(\bar k,\bar k')}{\sqrt{-h'(-\lambda)}},\qquad(67)$$

$$\left.\begin{aligned}&\langle N, \theta', \theta'' \,|\, R^{(3)} \,|\, V, \theta \rangle \\ = 2\pi\,\delta(\omega'+\omega''-\omega)\frac{g}{\sqrt{2V}}\frac{1}{2}&\left[\Phi_1(\bar{k}', \bar{k})\frac{f(\omega'')}{\sqrt{\omega''}} + \Phi_1(\bar{k}'', \bar{k})\frac{f(\omega')}{\sqrt{\omega'}}\right].\end{aligned}\right\} \tag{68}$$

It can be shown that the sum over all the roots in (65) corresponding to the scattering states in paragraph II and the last term of (64) can be expressed in terms of the matrix $R^{(3)}$. Using this, we can write (64) as

$$\left.\begin{aligned}&\langle V, \theta \,|\, R^{(1)} + R^{(1)*} + R^{(1)*} R^{(1)} \,|\, V', \theta' \rangle \\ -\langle V, \theta \,|\, R^{(2)*} R^{(2)} \,|\, V', \theta' \rangle &+ \langle V, \theta \,|\, R^{(3)*} R^{(3)} \,|\, V', \theta' \rangle = 0.\end{aligned}\right\} \tag{69}$$

It now follows that *the S-matrix of the Lee model* which, for the states considered in this paragraph, is given by

$$S = 1 + R^{(1)} + R^{(2)} + R^{(3)}, \tag{70}$$

is not unitary, because the probability for the transitions $V + \theta \rightarrow V_{-\lambda} + \theta'$ is to be counted negative in (69). As was suggested earlier, we see instead that the S-matrix has the property

$$\eta S^* \eta S = 1 \tag{71}$$

if the diagonal elements of η belonging to the states $|\, V_{-\lambda}, \theta \rangle$ are put equal to -1. It can also be shown that, if transitions *from* the states $|\, V_{-\lambda}, \theta \rangle$ are considered, a similar result will be obtained. The non-unitariness of the transformation (50) between the free-particle states and the physical states has its close correspondence in the non-unitariness of the S-matrix and makes the model unacceptable for physical reasons.

At this stage, one might ask if it is not possible to reinterpret the formalism with the aid of an argument similar to hole theory in quantum electrodynamics. One would then, *e. g.*, call the state $|\, V_{-\lambda} \rangle$ the vacuum, and the state which is here called the vacuum a state with one "anti-particle". However, it is easily seen that it is not possible to make the formalism consistent in this way as no reinterpretation along such lines will ever change the non-unitary properties of the S-matrix in (69).

The conclusion of our discussion is then that the model suggested by T. D. Lee is in accordance with the physical probability concept only if a cut-off is introduced and if the renormalized coupling constant is less than the critical value given by *Eq.* (34a). In this case, the constant N^2 lies between zero and one, as is expected from general arguments.[2] If there is no cut-off, the critical value of the coupling constant is zero.

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References.

(1) T. D. Lee, Phys. Rev. **95**, 1329 (1954).

(2) This was first shown by J. Schwinger (unpublished) and has since been found by several authors. Cf. H. Umezaha and S. Kamefuchi, Progr. Theor. Phys. **6**, 543 (1951); G. Källén, Helv. Phys. Acta **25**, 417 (1952); H. Lehmann, Nuovo Cimento **11**, 342 (1954); M. Gell-Man and F. E. Low, Phys. Rev. **95**, 1300 (1954). Appendix II of the paper by Lee[1] also contains a proof of this theorem.

(3) The "free-particle states" introduced in this way are of the same kind as the free-particle states used, *e.g.*, in the Tamm-Dancoff method, but entirely different from the so-called "incoming (or outgoing) free-particle states" used in other formulations of relativistic field theories. As will be seen below, the Tamm-Dancoff approach gives the exact solution for this model.

(4) G. Källén, Helv. Phys. Acta **25**, 417 (1952).

(5) An indefinite metric has earlier been used in quantum field theory by P. A. M. Dirac, Proc. Roy. Soc. **A 180**, 1 (1942) in a connection which is not too different from the one used here. Cf. also W. Pauli, Rev. Mod. Phys. **15**, 175 (1943). A result of R. P. Feynman, Phys. Rev. **76**, 749 (1949) particularly p. 756 implies the implicit use of an indefinite metric. Cf. W. Pauli, Prog. Theor. Phys. **5**, 526 (1950). An indefinite metric has also been used in quantum electrodynamics for a treatment of scalar photons. Cf. S. N. Gupta, Proc. Phys. Soc. **53**, 681 (1950) and K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

(6) C. Møller, Dan. Mat. Fys. Medd. **23**, no. 1 (1945); ibid. **22**, no. 19 (1946).

Appendix I.

In this appendix, we show by an explicit calculation how the indefinite metric is able to account for the negative sign on the right hand of the anticommutator

$$\left\{ \psi_V^+(\bar{p}), \psi_V(\bar{p}') \right\} = \delta_{\bar{p}, \bar{p}'} \frac{1}{N^2}. \tag{A.1}$$

We compute the vacuum expectation value of this quantity for $g > g_{\text{crit}}$ and $p = p'$, and obtain

$$\langle 0 | \{ \psi_V^+(\bar{p}), \psi_V(\bar{p}) \} | 0 \rangle = \sum_{|z\rangle} | \langle 0 | \psi_V(\bar{p}) | z \rangle |^2 \langle z | \eta | z \rangle. \tag{A.2}$$

In (A.2) the summation is performed over any complete set of states. We can, *e.g.*, sum over all physical states and get contributions from the physical V-particle state, the state $| V_{-\lambda} \rangle$, and the scattering states $| N, \theta \rangle$. According to the result of paragraph II, these contributions will be

$$\left. \begin{aligned} \langle 0 | \{ \psi_V^+(\bar{p}), \psi_V(\bar{p}) \} | 0 \rangle &= 1 + \sum_{\bar{k}} | \beta(\bar{k}) |^2 - \frac{1}{| h'(-\lambda) |} \\ &= 1 + \sum_{\bar{k}} | \beta(\bar{k}) |^2 + \frac{1}{h'(-\lambda)}. \end{aligned} \right\} \tag{A.3}$$

If there were no indefinite metric, the right-hand side would be positive and larger than one. This is also the usual proof[2)] that N^2 is a positive number less than one. In our case, the last term has a negative sign, and there is no general principle according to which the right-hand side of (A.3) has a definite sign. We shall now show explicitly that this quantity has the correct value given by *Eq.* (33). The proof is essentially based on the fact that the function $h(z)$ defined by (36) and extended to the complex plane by

$$h(z) = z\left[1 + \frac{g^2}{2V}\sum_{\bar{k}} \frac{f^2(\omega)\, z}{\omega^3(\omega - z)}\right] \tag{A.4}$$

has zeros only on the real axis. Indeed, one has with $z = x + iy$,

$$Im\,\frac{h(z)}{z} = \frac{g^2}{2V}\, Im \sum_{\bar{k}} \frac{f^2(\omega)\, z}{\omega^3(\omega - z)} = \frac{g^2}{2V}\sum_{\bar{k}} \frac{f^2(\omega)\, y}{\omega^2[(\omega - x)^2 + y^2]}, \tag{A.5}$$

which is always different from zero for $y \neq 0$.

Moreover, passing to the limit $V \to \infty$, $h(z)$ transforms into an analytic function given by

$$h(z) = z\left[1 + \gamma z \int_{\mu}^{\infty} f^2(\omega)\, \frac{\sqrt{\omega^2 - \mu^2}\, d\omega}{\omega^2(\omega - z)}\right] \tag{A.4 a}$$

$\left(\text{with the abbreviation } \gamma = \frac{g^2}{4\pi^2}\right)$ which is unique in the complex plane cut along the real axis from μ to positive infinity. The imaginary part of $h(z)$ is discontinuous at this part of the real axis, having opposite signs in the upper and the lower half plane, whilst the real part is continuous. To this ambiguity of $h(z)$ corresponds the circumstance that $z = \mu$ is a branching point of the square root type of $h(z)$ (*cf.* the explicit form given in Appendix II for the particular case $f(\omega) = 1$).

These properties of $h(z)$ enable us to evaluate the integral

$$\frac{1}{2\pi i}\int_{C} \frac{dz}{h(z)}$$

along the path illustrated in Fig. 1 in two different ways. We first remark that

$$\left.\begin{aligned} \sum_{\bar{k}} |\beta(\bar{k})|^2 &= \gamma \int_{\mu}^{\infty} f^2(\omega)\sqrt{\omega^2 - \mu^2}\, d\omega \left[h^2(\omega) + \left(\frac{\pi\gamma}{\omega} f^2(\omega)\sqrt{\omega^2 - \mu^2}\right)^2\right]^{-1} \\ &= \frac{1}{\pi} \lim_{\varepsilon \to 0} Im \int_{\mu}^{\infty} \frac{d\omega}{h(\omega - i\varepsilon)}. \end{aligned}\right\} \tag{A.6}$$

We now divide the path C into two parts. One of them, C_1, starts from a point $z = R - i\varepsilon$ with arbitrarily large R and arbitrarily

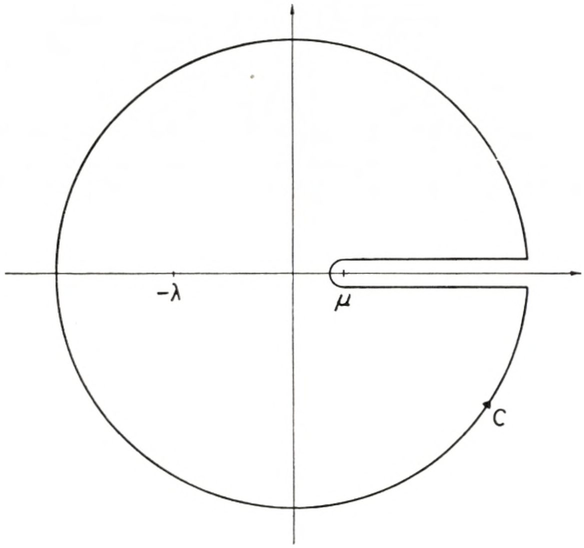

Fig. 1. The path C in *Eq.* (A. 9).

small, positive ε, goes below the real axis at a distance ε from it, encircles the point $z = \mu$ in the negative direction, returns above the real axis at a distance ε, and ends at the point $z = R + i\varepsilon$. The second part, C_R, is a large circle with radius R of which a small part near the positive real axis is omitted.

Performing the limiting process $\varepsilon \to 0$, in which the contribution of the circular arc of C_1 gets arbitrarily small, one first obtains

$$\lim_{\varepsilon \to 0} \int_{C_1} \frac{dz}{h(z)} = -2\, i \lim_{\varepsilon \to 0} Im \int_{\mu}^{\infty} \frac{dz}{h(z - i\varepsilon)} = -2\pi i \sum_{\bar{k}} \left|\beta(\bar{k})\right|^2. \quad \text{(A. 7)}$$

In this limit, the second part C_R of C goes over into the full circle C_R. The corresponding integral is easily evaluated with the aid of the asymptotic form of the function $h(z)$ (cf. the remarks before *Eq.* (43)) and gives

$$\int_{C_R} \frac{dz}{h(z)} = 2\pi i \frac{1}{N^2}. \quad \text{(A. 8)}$$

Hence, in this way we obtain

$$\frac{1}{2\pi i}\int_C \frac{dz}{h(z)} + \sum_{\bar{k}} |\beta(\bar{k})|^2 = \frac{1}{N^2}. \qquad (A.9)$$

On the other hand, the absence of non-real zeros of $h(z)$ and a knowledge of the residues of $h(z)^{-1}$ at the poles $z = 0$ and $z = -\lambda$ permits a direct evaluation of the integral

$$\frac{1}{2\pi i}\int_C \frac{dz}{h(z)} = 1 + \frac{1}{h'(-\lambda)}. \qquad (A.10)$$

Hence,

$$1 + \sum_{\bar{k}} |\beta(\bar{k})|^2 + \frac{1}{h'(-\lambda)} = \frac{1}{N^2}. \qquad (A.11)$$

Eqs. (A.11) and (A.3) together give the expected result (A.1). If the coupling constant is less than the critical value, the integrand in (A.9) will have no pole at $z = -\lambda$, and the last term in (A.10) will be missing. Other matrix elements of the commutators and anticommutators can be treated in similar ways.

Appendix II.

In the *particular case of no cut-off* $f(\omega) = 1$, $1/N = 0$ the function $h(z)$ (cf. (A.4a)) can be expressed in closed form:

$$\left.\begin{aligned} h(\omega \pm i\varepsilon) &= \omega + \gamma\left[\omega + \frac{\pi\mu}{2} - \sqrt{\omega^2 - \mu^2}\left(\log\frac{\omega + \sqrt{\omega^2 - \mu^2}}{\mu} \mp i\pi\right)\right] \\ &\text{if} \quad \omega > \mu \quad \text{and} \quad \varepsilon > 0, \end{aligned}\right\} \qquad (A.12)$$

$$h(-\lambda) = -\lambda + \gamma\left[-\lambda + \frac{\mu\pi}{2} + \sqrt{\lambda^2 - \mu^2}\log\frac{\lambda + \sqrt{\lambda^2 - \mu^2}}{\mu}\right] \quad \text{if} \quad \lambda > \mu. \qquad (A.13)$$

Apart from the imaginary part in (A. 12) these two cases can also be represented by the same fomula if an absolute value is taken for the argument under the logarithm. For the third interval of the real axis, one has

$$h(\omega) = \omega + \gamma \left[\omega - \sqrt{\mu^2 - \omega^2} \arcsin \frac{\omega}{\mu} + \frac{\pi}{2} \frac{\omega^2}{\mu + \sqrt{\mu^2 - \omega^2}} \right] \text{ if } -\mu < \omega < \mu. \quad (A.14)$$

These expressions can be used to find the position of the root

$$h(-\lambda) = 0 \quad (A.15)$$

both in the weak and in the strong coupling limit. For weak coupling, we find from (A.13)

$$\lambda \approx \frac{\mu}{2} e^{1/\gamma} \quad \text{if } \gamma << 1, \quad (A.16)$$

which *excludes any kind of power series expansion.** In the strong coupling limit the application of (A.14) gives the following expression for the root:

$$-\omega \equiv \lambda \approx \frac{4}{\pi} \frac{\mu}{\gamma} \quad \text{if } \gamma >> 1 \quad (A.17)$$

with a possibility of an expansion in powers of γ^{-1}.

* This is of some interest in connection with the failure to obtain a power series with a finite radius of convergence by application of perturbation methods to some examples of renormalizable field theories. Cf. C. A. HURST, Proc. Cambr. Phil. Soc. **48**, 625 (1952); W. THIRRING, Helv. Phys. Acta **26**, 33 (1953); A. PETERMANN, Phys. Rev. **89**, 1160 (1953), and R. UTIYAMA and T. IMAMURA, Prog. Theor. Phys. **9**, 431 (1953).

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